Multiscale Discrete Geometry

Mouhammad Said[1,2], Jacques-Olivier Lachaud[1], Fabien Feschet[2]

[1] LAMA, UMR 5127 CNRS, University of Savoie

[2] LAIC, University of Clermont-Ferrand

Reunion GEODIB 2010

28/06/2010

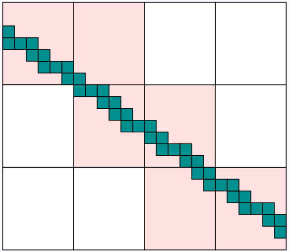
Introduction

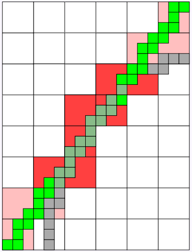
Objective

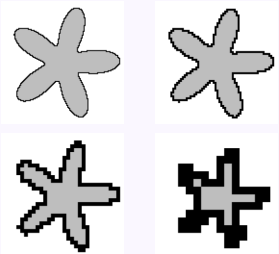
- Understanding the multiscale behaviour of digital shapes
- Analytical description, extract geometric characteristics

Contribution

1. Analytic description of subsampled standard digital lines (DL), and DSS.
2. Output-sensitive algorithms for recognizing digital straight segments included in known DL.
3. Output-sensitive algorithm for subsampling a digital shape boundary.







Extension of [Figueiredo99] to standard digital line

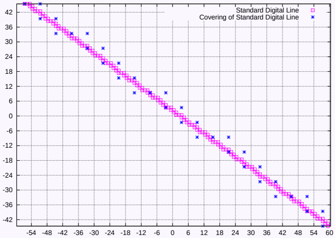
Theorem 1 (Second and Fourth Quadrants)

The digital straight line Δ of $S(h, v)$ covering the standard digital line $D(a, b, \mu)$ of $\mathbb{Z}^2$ is defined by :

$$-p + Q_2 - Q_1 + SI \leq \alpha X + \beta Y < Q_3 - Q_2 + SS$$

where $\alpha = \frac{ah}{g}$, $\beta = \frac{bv}{g}$, $g = gcd(ah, bv)$, $p = \alpha + \beta$, $Q_k = \left[\frac{(k-1)\mu + k(a+b) - 1}{g}\right]$, $R_k = \left\{\frac{(k-1)\mu + k(a+b) - 1}{g}\right\}$, $k = 1, 2, 3$.

$D(7, 9, 6)$ with
$(h, v) = (6, 6)$
$D : 6 \leq 7x + 9y < 15$
$\Delta : -12 \leq 7X + 9Y < 4$

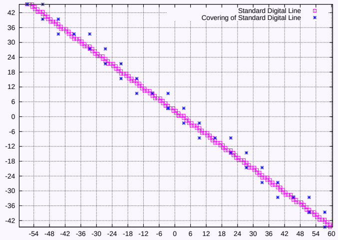


Theorem 2

The covering line Δ is standard.

$$\text{i.e. } (Q_3 - Q_2 + SS) - (-p + Q_2 - Q_1 + SI) = \alpha + \beta$$

$D(7,9,6)$ with
$(h,v) = (6,6)$
$D : 6 \leq 7x + 9y < 15$
$\Delta : -12 \leq 7X + 9Y < 4$



Extension of [Figueiredo99] to standard digital line

Theorem 3 (First and Third Quadrants)

The digital straight line Δ of $S(h, v)$ covering the standard digital line $D(a, b, \mu)$ of $\mathbb{Z}^2$ is defined by :

$$-p_1 + Q'_2 - Q'_1 + SI \leq \alpha X - \beta Y < p_2 + Q'_3 - Q'_2 + SS$$

where $\alpha = p_1 = \frac{ah}{g}$, $\beta = p_2 = \frac{bv}{g}$, $g = gcd(ah, bv)$, $p = \alpha + \beta$, $Q'_k = \left[\frac{(k-1)\mu+ka+b-1}{g}\right]$, $R'_k = \left\{\frac{(k-1)\mu+ka+b-1}{g}\right\}$, $k = 1, 2, 3$, and,

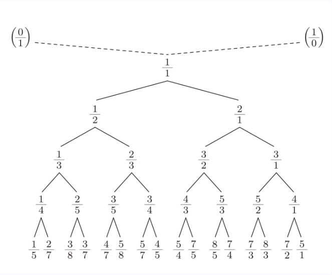
$$SI = \begin{cases} 0 & \text{if} \quad R'_2 \leq R'_1 \\ 1 & \text{otherwise} \end{cases} \qquad SS = \begin{cases} 0 & \text{if} \quad R'_3 \leq R'_2 \\ 1 & \text{otherwise} \end{cases}$$

What about digital segments ?

- Digital shape boundary made of digital segments not lines
- Hard to get an analytical description

Proposition 3

$$\begin{array}{ccc} \text{standard line } D & \xrightarrow{\text{subsampling}} & \text{standard line } D'(a', b', \mu') \\ \cup & & \cup \\ \text{standard segment } S & \longrightarrow & \text{standard segment } S' \\ & & \text{slope} = \text{ancestor of } \frac{a'}{b'} \text{ in SB tree} \end{array}$$



Stern-Brocot Tree, hierarchy containing all the positive irreductible rational fractions

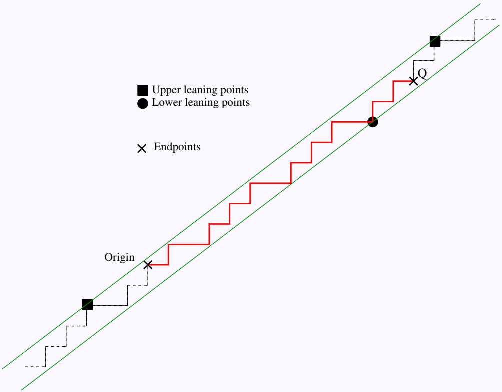
Problem

How to determine slope and shift of segment S' ?

A refinement algorithm to recognize segments

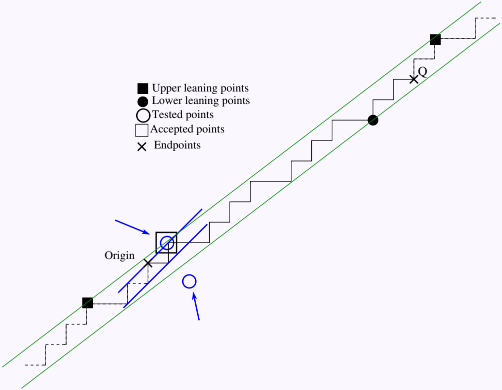
- we know that $S' \subset D'$, D' with known characteristics
- variant of DSS recognition algorithm **DR95** [Debled,Reveillès95]
- test only possible weakly exterior leaning points

characteristics of S',
$\subset D(13, 17, -5)$?



- we know that $S' \subset D'$, D' with known characteristics
- variant of DSS recognition algorithm **DR95** [Debled,Reveillès95]
- test only possible weakly exterior leaning points

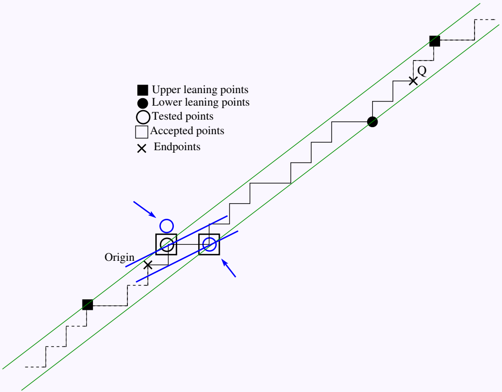
characteristics of S',
$\subset D(13, 17, -5)$?



DSS(0,1,0), $U'(1,1)$ and $L'(2,-1)$

- we know that $S' \subset D'$, D' with known characteristics
- variant of DSS recognition algorithm **DR95** [Debled,Reveillès95]
- test only possible weakly exterior leaning points

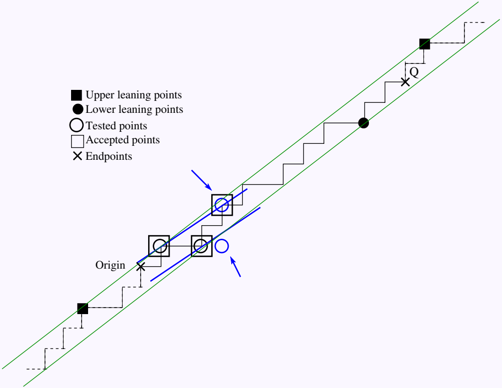
characteristics of S',
$\subset D(13, 17, -5)$?



DSS(1,1,-1), $U'(1,2)$ and $L'(3,1)$

- we know that $S' \subset D'$, D' with known characteristics
- variant of DSS recognition algorithm **DR95** [Debled,Reveillès95]
- test only possible weakly exterior leaning points

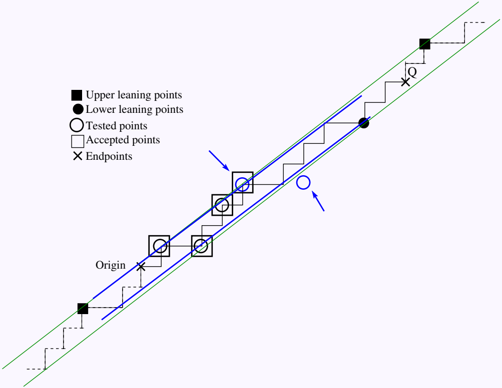
characteristics of S',
$\subset D(13, 17, -5)$?



$DSS(1, 2, -1)$, $U'(4, 3)$ and $L'(4, 1)$

- we know that $S' \subset D'$, D' with known characteristics
- variant of DSS recognition algorithm **DR95** [Debled,Reveillès95]
- test only possible weakly exterior leaning points

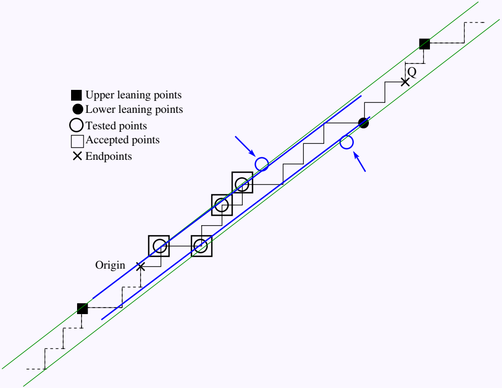
characteristics of S',
$\subset D(13, 17, -5)$?



$DSS(2, 3, -1)$, $U'(5, 4)$ and $L'(8, 4)$

- we know that $S' \subset D'$, D' with known characteristics
- variant of DSS recognition algorithm **DR95** [Debled,Reveillès95]
- test only possible weakly exterior leaning points

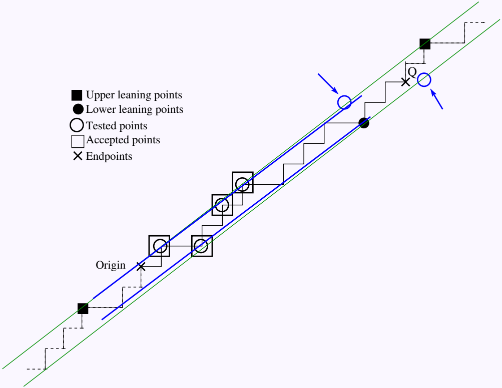
characteristics of S',
$\subset D(13, 17, -5)$?



$DSS(3, 4, -1)$, $U'(6, 5)$ and $L'(10, 6)$

- we know that $S' \subset D'$, D' with known characteristics
- variant of DSS recognition algorithm **DR95** [Debled,Reveillès95]
- test only possible weakly exterior leaning points

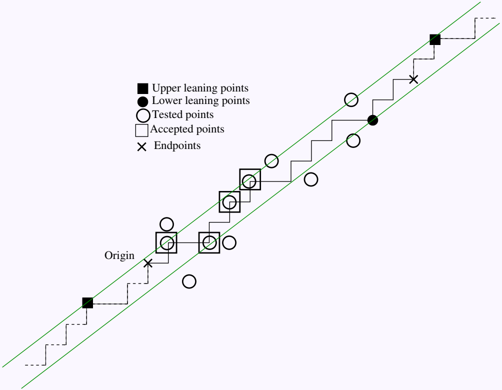
characteristics of S',
$\subset D(13, 17, -5)$?



$DSS(3, 4, -1)$, $U'(10, 8)$ and $L'(14, 9)$

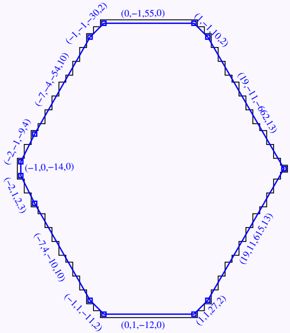
- we know that $S' \subset D'$, D' with known characteristics
- variant of DSS recognition algorithm **DR95** [Debled,Reveillès95]
- test only possible weakly exterior leaning points

characteristics of S', $\subset D(13, 17, -5)$?
$S'(3, 4, -1)$, 11 tested points

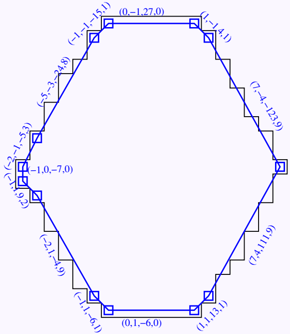


$DSS(3, 4, -1)$, $U'(10, 8)$ and $L'(14, 9)$

- Multiscale computation of the boundary of a digital shape, with 164 points, decomposed into 13 segments.



$(h,v) = (1,1)$
61 tested points and extracted DSS.



$(h,v) = (2,2)$
44 tested points and extracted DSS.

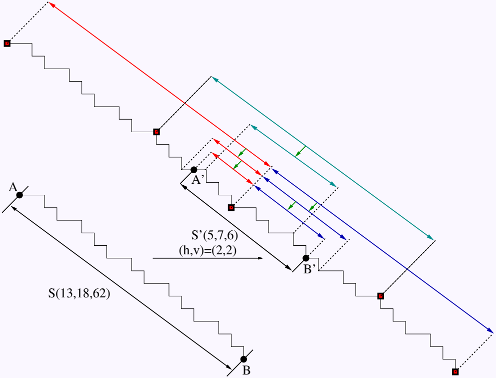
Complexity for multiscale computation of digital shape

Output-sensitive algorithm : depends on the coefficients of the continued fractions of the slope of output segments

A coarsening algorithm to recognize segments

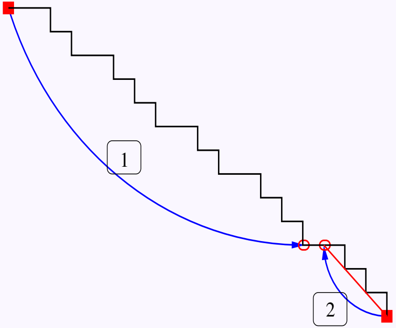
- we know that $S' \subset D'$, D' with known characteristics
- depends to the bottom-up moving in the Stern-Brocot Tree

characteristics of S',
$\subset D'(13, 18, 16)$?



- we know that $S' \subset D'$, D' with known characteristics
- depends to the bottom-up moving in the Stern-Brocot Tree

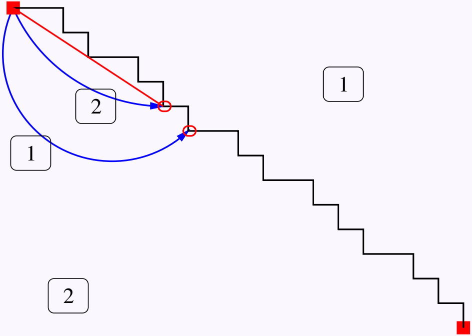
characteristics of S',
$\subset D'(13, 18, 16)$?
Complexity :
$O(n - n_1)$, where
$n = \sum_{i=0}^{n} u_n$ with
$\frac{13}{18} = [u_0, \ldots, u_n]$, and
$n_1 = \sum_{i=0}^{n_1} u_{n_1}$ with
$\frac{1}{1} = [u_0, \ldots, u_{n_1}]$.



Left Lower Slope, $S_1(1, 1)$

- we know that $S' \subset D'$, D' with known characteristics
- depends to the bottom-up moving in the Stern-Brocot Tree

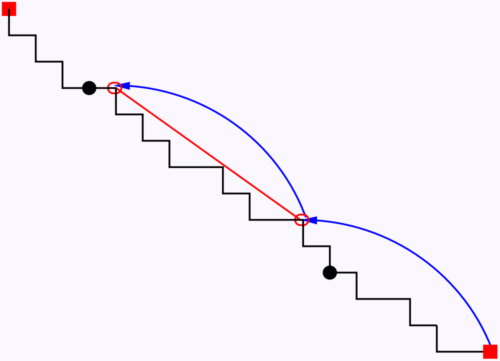
characteristics of S', $\subset D'(13, 18, 16)$?
Complexity :
$O(n - n_2)$, where $n_2 = \sum_{i=0}^{n_2} u_{n_2}$ with $\frac{2}{3} = [u_0, \ldots, u_{n_2}]$.



Right Lower Slope, $S_2(2, 3)$

- we know that $S' \subset D'$, D' with known characteristics
- depends to the bottom-up moving in the Stern-Brocot Tree

characteristics of S',
$\subset D'(13, 18, 16)$?
Complexity :
$O(n - n_3)$, where
$n_3 = \sum_{i=0}^{n_3} u_{n_3}$ with
$\frac{5}{7} = [u_0, \ldots, u_{n_3}]$.



Upper Slope, $S_3(5, 7)$

The slope of S' is :

$$\text{Slope of } S' = \text{DeepestSlope}(\ S_1,\ S_2,\ S_3\ ) = S_3 = (5, 7, 6).$$

The computational complexity of this algorithm is :

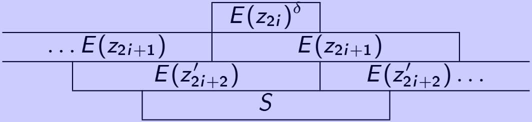
$$O(n - n'), \text{ where } n' = min\{n_1, n_2, n_3\}.$$

Standard Digital Lines Intersection

- Given a *DSS* S,
- Build two *DSL* whose intersection contains S (related to the downward moves in the Stern-Brocot tree),
- Its main connected part has the same arithmetic characteristics, same number of patterns.

Proposition 1 (Even Slope)

The main connected part S of the intersection between $E(z_{2i+1})$ with $z_{2i+1} = [0, u_1, \ldots, u_{2i}, \delta]$ and $E(z'_{2i+2})$ with $z'_{2i+2} = [0, u_1, \ldots, u_{2i} - 1, 1, \delta]$ is defined as their common part as placed below :



The word S is exactly $w_1 E(z_{2i})^{\delta} w_2$, with $w_1 = E(z_1)^{u_2} \cdots E(z_{2i-2k-1})^{u_{2i-2k}} \cdots E(z_{2i-3})^{u_{2i-2}} \; E(z_{2i-1})^{u_{2i}-1}$ and $w_2 = E(z_{2i-2})^{u_{2i-1}} \cdots E(z_{2i-2k})^{u_{2i-2k+1}} \cdots E(z_2)^{u_3} \; E(z_0)^{u_1}$.

Example : Even Slope

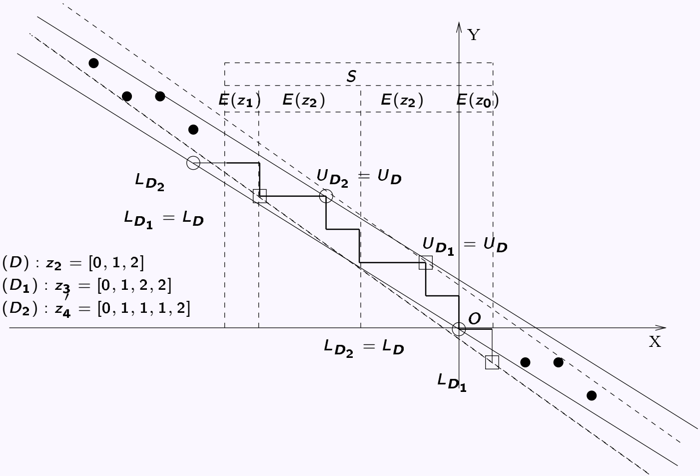
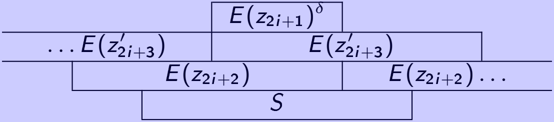


Fig.: Intersection of two patterns $E(z_3)$ and $E(z_4^{'})$, where S is the main connected part of their intersection.

Proposition 2 (Odd Slope)

The main connected part S of the intersection between $E(z_{2i+2})$ with $z_{2i+2} = [0, u_1, ..., u_{2i+1}, \delta]$ and $E(z'_{2i+3})$ with $z'_{2i+3} = [0, u_1, ..., u_{2i+1} - 1, 1, \delta]$ is defined as their common part as placed below :



The word S is exactly $w_1 E(z_{2i+1})^{\delta} w_2$, with $w_1 = E(z_1)^{u_2}\ E(z_3)^{u_4} \cdots E(z_{2i-2k-1})^{u_{2i-2k}} \cdots E(z_{2i-3})^{u_{2i-2}}\ E(z_{2i-1})^{u_{2i}}$. and $w_2 = E(z_{2i})^{u_{2i+1}-1} E(z_{2i-2})^{u_{2i-1}} E(z_{2i-4})^{u_{2i-3}} \cdots E(z_{2i-2k})^{u_{2i-2k+1}} \cdots E(z_2)^{u_3} E(z_0)^{u_1}$.

Example : Odd Slope

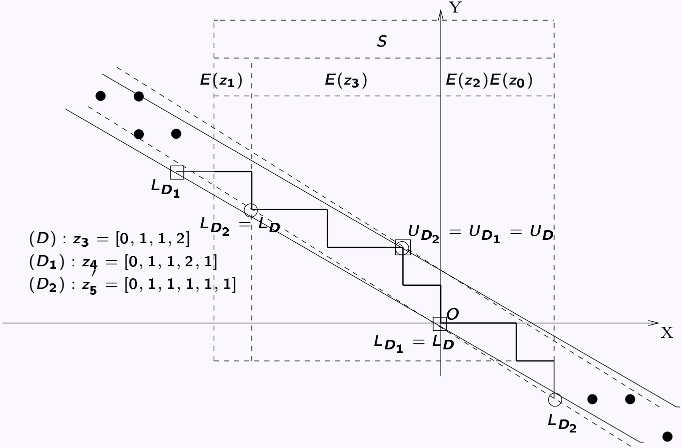


Fig.: Intersection of two patterns $E(z_4)$ and $E(z_5^{'})$, where S is the main connected part of their intersection.

The coordinates of the lower leaning points of D_1 and D_2 of remainders :

- $\delta\mu + \mu - \delta$ and $\delta\mu$ respectively in the even complexity, or
- $\delta\mu$ and $\delta\mu + \mu - \delta$ respectively in the odd complexity, are given by :

	D_1	D_2
D has an even slope	$(\mu(b - q_{2i-1}) - \delta q_{2i}, \mu(p_{2i-1} - a) + \delta p_{2i}) + k(-\delta q_{2i} - q_{2i-1}, \delta p_{2i} + p_{2i-1})$	$(\mu(b - q_{2i-1}), \mu(p_{2i-1} - a)) + k(-(\delta + 1)q_{2i} + q_{2i-1}, (\delta + 1)p_{2i} - p_{2i-1})$
D has an odd slope	$(\mu q_{2i}, -\mu p_{2i}) + k(-\delta q_{2i+1} - q_{2i}, \delta p_{2i+1} + p_{2i})$	$(\mu q_{2i} - \delta q_{2i+1}, -\mu p_{2i} + \delta p_{2i+1}) + k(-(\delta + 1)q_{2i+1} + q_{2i}, (\delta + 1)p_{2i+1} - p_{2i})$

Proposition 3

Let $D_1(a_1, b_1, \mu_1)$ and $D_2(a_2, b_2, \mu_2)$ be two standard DSL of slopes $\frac{a_1}{b_1} = [0, u_1, ..., u_n, \delta]$ and $\frac{a_2}{b_2} = [0, u_1, ..., u_n - 1, 1, \delta]$, where n=2i even or n=2i+1 odd, $\mu_1 = \delta\mu + (2i + 1 - n)(\mu - \delta)$ and $\mu_2 = \delta\mu + (n - 2i)(\mu - \delta)$. Then their main connected part is a DSS of slope z_n with δ patterns and shift μ.

- Input : $D_1(a_1, b_1, \mu_1)$ and $D_2(a_2, b_2, \mu_2)$
- Using Theorem 1 to compute the characteristics of Δ_1 and Δ_2.

$$-p^1 + Q_2^1 - Q_1^1 + SI^1 \leq a_1' X + b_1' Y < Q_3^1 - Q_2^1 + SS^1 \quad (\Delta_1)$$
$$-p^2 + Q_2^2 - Q_1^2 + SI^2 \leq a_2' X + b_2' Y < Q_3^2 - Q_2^2 + SS^2 \quad (\Delta_2)$$

Theorem

The digital intersection of two digital straight lines Δ_1 and Δ_2 of $S(h, v)$ covering respectively the two digital straight lines D_1 and D_2 of $\mathbb{Z}^2$ is defined by :

$$A^1 \leq X' < B^1 \tag{1}$$

The expression of the boundaries of Y' depends on the sign of λ_2 :

$$\lambda_2 > 0, -\left[\frac{-A^2 + \lambda_1 X'}{\lambda_2}\right] \leq Y' < -\left[\frac{-B^2 + \lambda_1 X'}{\lambda_2}\right] \tag{2}$$
$$\lambda_2 < 0, \left[\frac{B^2 - \lambda_1 X'}{\lambda_2}\right] + 1 \leq Y' < \left[\frac{A^2 - \lambda_1 X'}{\lambda_2}\right] + 1 \tag{3}$$

Example

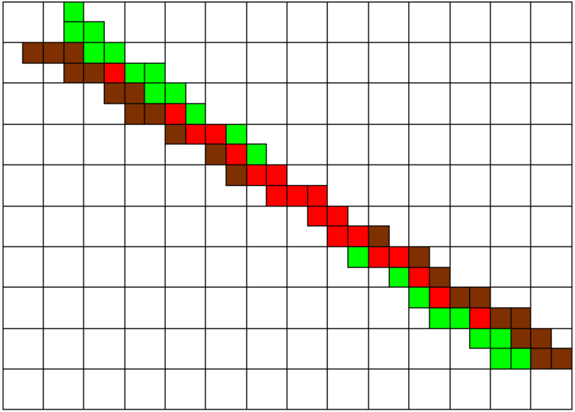


Fig.: Intersection of $D_1(3,4,3)$ drawn as light green boxes and $D_2(3,5,2)$ drawn as brown boxes, their intersection is drawn by red boxes.

Example

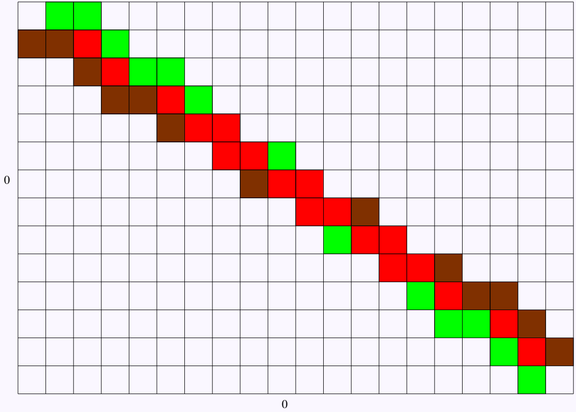


Fig.: Intersection of $\Delta_1(3,4,-2)$ drawn as light green boxes and $\Delta_2(3,5,-3)$ drawn as brown boxes, and their intersection is drawn by red boxes.

Theorem 4

Let $n_{IP}(\Delta_1, \Delta_2)$ be a number of intersection points of Δ_1 and Δ_2 and $n_P(X')$ be the number of points who have the same abscissa X'. Then,

$$n_{IP}(\Delta_1, \Delta_2) = \sum_{X'=A^1}^{B^1-1} n_P(X')$$

where,

- If $\lambda_2 > 0$, then,

$$n_P(X') = \begin{cases} \left\lfloor \dfrac{a_2' + b_2'}{\lambda_2} \right\rfloor & \text{if} \quad R_1 \geq R_2 \\ \left\lfloor \dfrac{a_2' + b_2'}{\lambda_2} \right\rfloor + 1 & \text{otherwise} \end{cases}$$

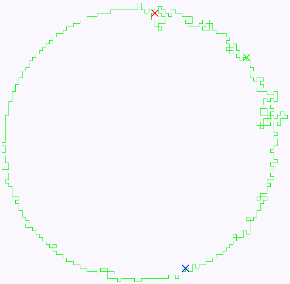
- If $\lambda_2 < 0$, then,

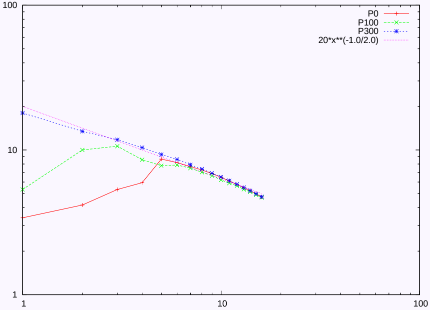
$$n_P(X') = \begin{cases} \left\lfloor \dfrac{-a_2' - b_2'}{\lambda_2} \right\rfloor & \text{if} \quad R_1 \geq R_2 \\ \left\lfloor \dfrac{-a_2' - b_2'}{\lambda_2} \right\rfloor + 1 & \text{otherwise} \end{cases}$$

with $R_1 = \left\{ \dfrac{-A^2 + \lambda_1 X'}{\lambda_2} \right\}$ and $R_2 = \left\{ \dfrac{-B^2 + \lambda_1 X'}{\lambda_2} \right\}$.

Scale Profile of a point p on a digital contour

Scale Profile : $P_n(p) = \text{sequence}(log i, log(E(L^{\mu_i}))_{i=1...n}$, where E is the average operator, L^{μ_i} are the digital lengths of the maximal segments covering p for all thickness ν_i.





Thank you for your attention

Conclusion

- Presented new results about the covering of discrete objects by regular tilings (h, v).
- Presented a novel fast DSS recognition algorithm.
- computational complexity is $\Theta(\sum_{i=0}^{k} u_k)$.
- Compute the exact multiscale covering of a digital contour in a time proportional to $M \times \overline{U}$, where $\overline{U}$ is the average of the partial quotient sum of the *output* subsampled DSS
- In most cases, this is clearly sublinear, and at worst, linear in the size of the contour.

Perspective

- This work is a first step towards the multi-scale computation of the tangential cover, a fundamental representation of digital curves.

Action **SmartDSS**(In D, In P, Q, Out S);
begin
  while *inside and* $p_k \neq \alpha$ do
    $(a, b) \leftarrow (p_k, q_k)$;
1   $(b', a') \leftarrow$ **Bézout**(p, q, k) /*$ab' - ba' = 1$*/;
    $U' \leftarrow U + (b - b', a - a')$;
    $L' \leftarrow L + (b', a')$;
    $\delta \leftarrow 1$, $loop \leftarrow 0$;
2   repeat
      $U' \leftarrow U' + (b, a)$, $L' \leftarrow L' + (b, a)$;
      if $U'_y \leq Q_y$ *and* $U' \in D$ then $loop \leftarrow 1$; break;
      else if $L'_x \leq Q_x$ *and* $L' \in D$ then $loop \leftarrow 2$; break;
      $\delta \leftarrow \delta + 1$;
    until $U'_y \geq Q_y$ *or* $L'_x \geq Q_x$;
3   if $loop = 1$ /**Increase slope with weak upper leaning point* U'*/ then
      UpdateSlope(*true*, δ, k, u, p, q);
4     $L \leftarrow L' - (b', a')$;
      if *not lul* then $L \leftarrow L - (b, a)$;
      ulu $\leftarrow$ *true*, *lul* $\leftarrow$ *false* ;
5   if $loop = 2$ /**Decrease slope with weak lower leaning point* L'*/ then
      UpdateSlope(*false*, δ, k, u, p, q);
6     $U \leftarrow U' - (b - b', a - a')$;
      if *not ulu* then $U \leftarrow U - (b, a)$;
      ulu $\leftarrow$ *false*, *lul* $\leftarrow$ *true* ;
    else
      inside $\leftarrow$ *false*
    ;
  $a \leftarrow p_k$, $b \leftarrow q_k$, $\mu \leftarrow aU_x - bU_y$;
end

Algorithm 1: The computation time is linear with the sum $\sum_{i=0}^{k} u_k$ with $\frac{a}{b} = [u_0, \ldots, u_k]$.